# The viscosity bound conjecture and hydrodynamics of M2-brane theory at finite chemical potential

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ABSTRACT: Kovtun, Son and Starinets have conjectured that the viscosity to entropy density ratio $\eta/s$ is always bounded from below by $\hbar/(4\pi k_B)$ for all forms of matter. The proposed viscosity bound appears to be saturated in all computations done whenever a supergravity dual is available. We consider the near horizon limit of a stack of M2-branes in the grand canonical ensemble at finite R-charge densities, corresponding to non-zero angular momentum in the bulk. The corresponding four-dimensional R-charged black hole in Anti-de Sitter space provides a holographic dual in which various transport coefficients can be calculated. We find that the shear viscosity increases as soon as a background R-charge density is turned on. We compute numerically the first few corrections to the shear viscosity to entropy density ratio $\eta/s$ and discover that up to fourth order all corrections originating from a non-zero chemical potential vanish, leaving the bound saturated. This is a sharp signal in favor of the saturation of the viscosity bound for event horizons even in the presence of some finite background field strength. We discuss implications of this observation for the conjectured bound.

JHEP10(2006)083

# Contents

## 1. Introduction

In recent years, string theory/gauge theory dualities have been proven to be invaluable tools in understanding strong coupling dynamics of certain quantum field theories. For any strong-weak type duality, computations that are possible on both sides of the duality are scarce. In these cases, taking various limits of both sides of the duality comes in handy, and may provide evidence for the duality. Previous examples of this include the PP-wave limit of the AdS/CFT correspondence [1], among others. The *hydrodynamic limit* could be one such simplifying yet non-trivial limit, where even strongly coupled quantum mechanical systems behave simply and universally. Hydrodynamics appears to be relevant to achieving a better understanding of the quark-gluon plasma (QGP) state in heavy ion collisions at RHIC. Studying near equilibrium phenomena in a hot, strongly coupled QCD plasma is never easy, even on the lattice. Extracting retarded Green's functions from Euclidean lattice computations requires long Minkowski time separations, which calls for a large number of lattice points [2]. Therefore using a dual description for QCD-like theories in order to extract transport coefficients may even find practical use in studying near-equilibrium QCD physics.

Sometime ago, Policastro, Son and Starinets proposed a prescription for calculating Minkowskian field theory Green's functions using the supergravity dual [11]. Since then an extensive study has been done where, using this prescription, various transport coefficients were calculated from the gravity side corresponding to the D-brane world-volume theories [3] as well as M2 and M5-brane theories in M-theory [4]. In [5], it was conjectured that the ratio of shear viscosity to entropy density is bounded from below

$$\frac{\eta}{s} \geq \frac{\hbar}{4\pi k_B} = 6.08 \times 10^{-13} K.s, \tag{1.1}$$

where $\eta$ is the shear viscosity and $s$ is the entropy density. The bound is well satisfied for weakly coupled systems, which could be understood intuitively by noticing that in a weakly coupled plasma, the mean free time for the constituents is long. The puzzling feature shared by every transport coefficient calculation performed so far [8] is that for all known holographic duals to various supersymmetric gauge theories at finite temperature, the proposed bound appears to be *saturated* [3]. This suggests that, at the *infinite coupling* limit where the supergravity description is adequate, there exists some sort of *universality* in the hydrodynamic description of all of these field theories (their dual being some black hole in anti-de Sitter space) [6]. It was also emphasized [6] that the universal nature of the ratio is connected to the universality of the black hole absorption cross section for low energy graviton scattering, at least for the cases where the AdS near horizon region has a flat space completion.

Just on dimensional grounds, the bound itself appears to be in harmony with the observation that $\eta/s$ is a product of the energy per effective degree of freedom in the field theory at a t'Hooft coupling $g_{YM}^2 N \gg 1$ and a time scale associated with the mean free flight time of the quasi-particle excitation [6]. According to the Heisenberg uncertainty principle, this product must be bounded from below by a multiple of $\hbar$ in order for the notion of quasi-particle to make sense. In [7], the universal nature of the ratio was further established and extended to a large class of supergravity backgrounds where the dual possesses a translationally invariant horizon. The entire class *saturates* the bound.

There is no known example where, at the infinite coupling limit, the bound holds but *isn't saturated*. One major motivation behind this work was to investigate whether there exist example(s) in which, at the strict infinite coupling limit, the viscosity bound is satisfied but not saturated. A particular setup which could potentially avoid the no-go theorems discussed in [7], involves supersymmetric gauge theories living on the world-volume of type II D-branes or membrane and five-brane theories in M-theory at finite global charge densities. From a lower dimensional prospective this corresponds to black holes with some finite gauge field strength turned on at the horizon. It is natural to believe that the hydrodynamic properties of these horizons with a non-zero gauge field must be different from the "*neutral horizons*".

Here we work in the grand canonical description at finite chemical potential corresponding to finite R-charge i.e., $\langle j^0 \rangle \neq 0$ where $j^0$ is the R-charge. Gravitationally, the R-charge arises from finite transverse rotation from a 10 or 11-dimensional point of view.

We find that turning on a finite R-charge background *increases* the viscosity. In the conclusion section we speculate as to why this happens. To our surprise, we discover that the viscosity to entropy density ratio $\eta/s$, remains the same as that for zero chemical potential case i.e., $1/(4\pi)$ up to forth order in powers of $\Omega/T_H$, where $\Omega$ is the angular velocity and $T_H$ is the Hawking temperature. This provides an evidence for the saturation of the bound even for the horizons with a finite gauge field.

The paper is organized as follows. We begin with a lightning review of the hydrodynamic limit in systems with many degrees of freedom. After reviewing the prescription for computing Minkowskian retarded Green's functions in section 3, we discuss the relevant gravitational background, i.e., R-charged Anti- de Sitter black hole in section 4. Gravitational perturbation theory of AdS$_4$ black holes is the subject of discussion in section 5. In section 6, we consider $\langle T_{\mu\nu} T_{\rho\sigma} \rangle$ correlation in Minkowskian signature using the prescription reviewed in section 3. Finally, we proceed to calculate the transport coefficient of interest here namely the shear viscosity $\eta$ and $\eta/s$ in section 7.

## 2. Hydrodynamic Limit of Quantum Field Theories

Linear response theory is the mathematical theory of the relaxation of small disturbances around equilibrium where the thermally averaged Minkowskian Green's functions (retarded, in order to account for causality) of the unperturbed system in $d$-dimensions fully characterize the system's response to the external stimuli

$$G^R_{\mu_1...\mu_j...}(\vec{q}) = \int d^d x e^{-iq.x} \theta(t) \langle [\widehat{Q}_{\mu_1...}(x), \widehat{Q}_{\mu_j...}(0)] \rangle_\beta. \tag{2.1}$$

Here, $G^R_{\mu_1...\mu_j...}(q)$ denotes the retarded Green's function, $Q_{\mu_1...}(x)$ is the operator corresponding to the conserved current which couples to the external world disturbance, $\mu_i$s are some spacetime indices and $\beta$ is the inverse temperature. It is argued, and in simple cases explicitly shown, that the slowly varying (both in space and time) behavior of the Minkowskian Green's functions of interacting field theories has a specific pole structure imposed by the "hydrodynamic equations". These hydrodynamic limit conditions are usually satisfied when local thermal equilibrium is achieved. To create such circumstances, a fluid must be in its high collision regime where interactions are important to the dynamics. Hydrodynamics is the study of small, long wavelength and low frequency fluctuations of a medium in the vicinity of its equilibrium point.[1] In this limit all the fine structure of cutoff-scale physics gets wiped out, leaving only a few transport coefficients at low energies and long distances. It turns out that the relevant degrees of freedom to a hydrodynamical description are the charge densities associated with various global symmetries at the UV cutoff scale, along with the phase of the order parameters if any phase transition exists [9]. This makes charge densities the only relevant degrees of freedom in the hydrodynamic limit. Transport coefficients appearing in these sets of equations are not themselves part of the hydrodynamic description but, rather, are inputs. These coefficients could, in principle, be

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[1]Clearly, in a conformal theory at finite temperature, the temperature provides the required length scale for the system.

calculated from the zero frequency limit of the two point Green's functions like (2.1). It is exactly this type of computations which we will be concerned with in this paper.

A fluid in equilibrium has a spatial energy-momentum tensor of the form $T_{ij} = p\delta_{ij}$, where $p$ is the pressure. Slightly away from equilibrium, extra stresses will be present as a result of viscous forces. The viscous part of the energy-momentum tensor of a fluid is proportional to the symmetric-traceless combinations of the momentum gradient of the fluid. In 2+1 dimensions, it takes the following form

$$T_{ij} - p\delta_{ij} = -\frac{1}{\epsilon + p}\left[\eta(\partial_i T_j^0 + \partial_j T_i^0 - \frac{2}{3}\delta_{ij}\partial_k T^{0k}) + \zeta\delta_{ij}\partial_k T^{0k}\right] + (T_{00} - \epsilon)\frac{\partial p}{\partial \epsilon}\delta_{ij}, \quad (2.2)$$

where $\epsilon = \langle T_{00}\rangle$. Also $\eta$ denotes the shear viscosity which is microscopically inversely proportional to the mean scattering rate. It is directly proportional to the mean free path for the effective degrees of freedom at a given value of the coupling. A strongly coupled fluid will have less viscosity as the mean free path is small and the energy in the perturbation, taking the system away from its equilibrium, gets redistributed among degrees of freedom very quickly. This simply means that strongly coupled fluids are better approximations to the hypothetical notion of an "*ideal fluid*".

Also, $\zeta$ is the bulk viscosity. In a conformal theory, $\zeta = 0$. The linearized hydrodynamic equations are given by

$$\partial_0 T^{0i} + \partial_j(T^{ij} - p\delta^{ij}) = 0, \quad (2.3)$$

$$\partial_0(T^{00} - \epsilon) + \partial_i T^{0i} = 0. \quad (2.4)$$

These equations along with (2.2) form the complete set of the hydrodynamic equations specifying the system in its hydrodynamic limit. The eigenfrequency of the system specified by (2.2), (2.3) and (2.4), which we will be focusing on in this paper, is called the shear mode and is shown to possess the following dispersion relation

$$\omega = -i\mathcal{D}q^2, \quad (2.5)$$

where $\mathcal{D} = \eta/(\epsilon + p)$ is called diffusion constant, $\epsilon$ is the internal energy density of the system and $p$ is the pressure. Noticing that the dispersion relation (2.5) appears as the pole structure of $G^R(\vec{q})$ in the hydrodynamic limit, enables us to read off the transport coefficient $\eta$, provided the hydrodynamic limit of $G^R(\vec{q})$ as well as the energy density and the pressure are known.

## 3. AdS/CFT in Minkowski Signature

In order to explore the hydrodynamic limit, one has to find a way to calculate the two point functions in Minkowskian signature. The celebrated Anti-de Sitter/Conformal Field Theory duality [10] is naturally formulated in Euclidean signature where the boundary configuration of the on-shell closed string background in the bulk of AdS acts as a source

for generating all the Green's functions of the corresponding boundary operator. Formally, one can write

$$\langle \mathcal{O}(x_{i_1})\mathcal{O}(x_{i_2})\ldots\mathcal{O}(x_{i_n})\rangle \propto \frac{\delta^n S_{cl}[\Phi_{cl}(x)]}{\delta\Phi_B(x_{i_1})\ldots\delta\Phi_B(x_{i_2})\delta\Phi_B(x_{i_n})}, \qquad (3.1)$$

where $\Phi_B(x_i)$ represents the closed string boundary value associated to the operator $\mathcal{O}(x)$ in the boundary theory and $S_{cl}$ is the on-shell action. Calculating Minkowskian Green's functions is problematic due to the characteristics of the boundary value problem for hyperbolic operators in Minkowskian signature AdS spacetime. The authors in [11] proposed a prescription for computing the Minkowskian thermal correlators in the boundary theory. According to this prescription

$$G_R(\vec{q}) = -2\mathcal{F}(u,\vec{q})|_{u=u_{boundary}}, \qquad (3.2)$$

where $u$ is the radial coordinate in AdS to be defined later, $\vec{q}$ is the momentum on the boundary, and the imaginary part of $\mathcal{F}$ is the Fourier component of the flux associated with the corresponding AdS fluctuation. Moreover only solutions which behave like an incoming wave at the horizon are kept. The prescription was further examined and confirmed to reproduce the desired and known results for the two examples worked out in [11] i.e., zero temperature $\mathcal{N}=4$ SYM theory in four dimensions and 2-dimensional finite temperature CFT dual to the BTZ black holes in $AdS_3$.

## 4. Gravitational Background

### 4.1 $D=4$, $\mathcal{N}=2$ U(1)$^4$ Extended Gauged Supergravities

Extended gauged supergravities arise as Kaluza-Klein reductions of both D=10 and D=11 supergravities. Amongst these compactifications, an $S^7$ reduction of D=11 supergravity to N=8 SO(8) gauged supergravity in D=4 admits a consistent truncation where only gauge fields in the Cartan subalgebra of the gauge group SO(8), i.e., a $U(1)^4$ subgroup of the gauge group, survive. This truncation allows for 4-charge AdS black holes in four dimensions. Such a consistent truncation of D=4, N=8 SO(8) supergravity with $U(1)^4$ gauge group includes four commuting gauge fields, three scalars, and three axions and is called minimal N=2, $U(1)^4$ supergravity. The Lagrangian for N=2 minimal supergravity is given by

$$e^{-1}\mathcal{L} = R - \frac{1}{2}(\partial\vec{\phi})^2 + 8g^2(\cosh\phi_1 + \cosh\phi_2 + \cosh\phi_3) \qquad (4.1)$$
$$-\frac{1}{4}\sum_{i=1}^{4} e^{\vec{a_i}\cdot\vec{\phi}}(F^i_{(2)})^2.$$

where the $\phi_i$'s are 3 scalars, $g$ is the inverse $AdS_4$ radius. The $\vec{a}_i$ are introduced in [12]. It can be shown that any non-axionic solutions to this minimal supergravity can be uplifted to 11-dimensional supergravity using the ansatz presented in [12].

## 4.2 4-Charge AdS$_4$ Black Holes

The near horizon geometry of the non-extremal M2-brane background which allows for up to four independent angular momenta (the dimension of the Cartan subalgebra of SO(8)) is given by [12]

$$ds_{11}^2 = \tilde{\Delta}^{2/3} \left[ -(H_1 H_2 H_3 H_4)^{-1/2} f dt^2 + (H_1 H_2 H_3 H_4)^{1/2} (f^{-1} dr^2 + r^2 d\vec{y}.d\vec{y}) \right] \quad (4.2)$$

$$+ g^{-2} \tilde{\Delta}^{-1/3} \sum_{i=1}^{4} X_i^{-1} (d\mu_i^2 + \mu_i^2 (d\phi_i^2 + g A_t^i dt)^2),$$

where

$$f = -\frac{\mu}{r} + 4g^2 r^2 H_1 H_2 H_3 H_4, \quad (4.3)$$

$$H_i = 1 + \frac{l_i^2}{\tilde{r}^2}, i = 1, 2, 3, 4,$$

$$X_i = H_i^{-1} (H_1 H_2 H_3 H_4)^{1/4},$$

$$A_t^i = \frac{1 - H_i^{-1}}{g l_i \sinh \alpha},$$

$$\tilde{\Delta} = \sum_{i=1}^{4} X_i \mu_i^2,$$

$$r = \frac{1}{2} (2m \sinh^2 \alpha)^{-1/6} \tilde{r}^2,$$

$$g^2 = (2m \sinh^2 \alpha)^{-1/3},$$

where the $\mu_i$'s are coordinates on the unit 3-sphere and $l_i$ are four angular momentum parameters. The background (4.2) gives rise to a duality between $AdS_4 \times S^7$ and the M2-brane theory at finite temperature and finite R-charge, such that in thermal equilibrium $\langle j^0 \rangle \neq 0$, where $j^\mu$ represents the R-charge current in the boundary theory.

The near horizon limit of the rotating M2-branes background (4.2) under an $S^7$ reduction gives rise to the four dimensional, 4-charge AdS black holes. As a result, it is more convenient to work within the framework of N=2 minimal supergravity described above while bearing in mind that any solution to this theory is an M-theory solution. Therefore we will focus only on constructing minimal supergravity perturbations.

To simplify the setup without losing much of the generality, we choose to work with black holes with four equal charges which further simplifies the N=2 minimal supergravity to just Einstein gravity with a cosmological constant coupled to four Maxwell fields in four dimensions. In fact, setting the four charges equal makes the dilaton sector decouple and partially simplifies the system without destroying its essential features.

AdS$_4$ black holes with four equal charges in this theory, with the horizon geometry being a space of constant curvature, are given by [12]

$$ds_4^2 = -H^{-2} f dt^2 + H^2 (f^{-1} dr^2 + r^2 d\Omega_{2,k}^2), \quad (4.4)$$

$$f = k - \frac{\mu}{r} + 4g^2 r^2 H^4,$$

$$H = 1 + \frac{\mu \sinh^2 \beta}{kr},$$

$$A_t^i = \sqrt{k}(1 - H^{-1})\coth\beta, i = 1, 2, 3, 4,$$

where $k = -1, 0, 1$ refers to the curvature of the horizon geometry.[2]

The case $k = 0$ needs special treatment and the result is the same background except the gauge field and the function $H$ are now changed to

$$H = 1 + \frac{\mu \sinh^2 \beta}{r}, \tag{4.5}$$
$$A_t^i = \frac{1 - H^{-1}}{\sinh\beta}, i = 1, 2, 3, 4.$$

The relation between the eleven and four dimensional metric is

$$l_i^2 g = 2\mu \sinh^2 \beta_i, \tag{4.6}$$
$$\sinh\beta_i = gl_i \sinh\alpha,$$
$$\mu = mg^5.$$

In what follows, we choose to work with the flat case $k = 0$. Upon uplifting to M-theory, this gives rise to the decoupling limit of the flat world-volume (parameterized by coordinates on $\Omega_{2,0}$) rotating M2-branes with all four possible rotation parameters going. Note that the metric now is written as

$$ds_4^2 = -H^{-2}fdt^2 + H^2(f^{-1}dr^2 + r^2(d\theta^2 + d\phi^2)), \tag{4.7}$$

where $\theta$ and $\phi$ are the dimensionless angular variables. [3]

Let us introduce a more commonly used [4] "$u$" coordinate for later use

$$r = \frac{R_0^2}{u}, \tag{4.8}$$
$$R_0^6 = \frac{\mu}{4g^2},$$

where $u$ is the new membrane radial coordinate. Note that functions $f$ and $H$ appearing in (4.4) are now written as

$$f = \frac{4g^2 R_0^4}{u^2}(H^4 - u^3), \tag{4.9}$$
$$H = 1 + \frac{\mu \sinh^2 \beta u}{R_0^2}.$$

Here, we record a few quantities associated to the background (4.2) for later usage.

A dimensionless combination $y = \mu \sinh^2 \beta / R_0^2$ will make an appearance later on. The angular velocity $\Omega$ corresponding to (4.2) is proportional to the chemical potential for the R-charge as viewed from the dual boundary theory. To calculate $\Omega$, we need to rewrite the

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[2]Note that in spacetimes with AdS$_4$ asymptotics, horizon topology is not restricted to just 2-spheres:the horizon manifold could be either of the 3 possible spaces of constant curvature.

[3]The prescription for switching to the dimensionful coordinates $x,y$ used in [4] is to notice that $2gx = \theta$ and $2gy = \phi$.

M-theory embedding of our $AdS_4$ black holes with four equal charges namely (4.2)[12]

$$ds_{11}^2 = -H^{-2}f dt^2 + H^2(f^{-1}dr^2 + r^2 d\vec{y}.d\vec{y}) + \sum_{i=1}^{4}(d\mu_i^2 + \mu_i^2(d\phi_i + gA_t^i dt)^2). \quad (4.10)$$

Using definition of $\Omega$

$$\Omega_i = -\frac{g_{t\phi_i}}{g_{\phi_i\phi_i}}, \quad (4.11)$$

where "$i$" labels each of the four independent angular velocities, one obtains

$$\Omega_i = \Omega = -gA_t^i = -g\frac{1 - H^{-1}}{\sinh\beta}. \quad (4.12)$$

The event horizon is located where $f = 0$. The horizon radius can be expressed as a power series in $\sinh\beta$. We will keep terms only up to forth order in $\sinh\beta$ (or equivalently $y^2$)

$$u_H = u_0 = 1 + \frac{4}{3}y + 2y^2 + \mathcal{O}(y^3). \quad (4.13)$$

The Hawking temperature associated with this horizon is given by

$$T_H = \frac{\partial_r(H^{-2}f)}{4\pi}|_{r=r_H}, \quad (4.14)$$

which leads to

$$T_H = 3\frac{2^{-2/3}}{\pi}(\mu g^4)^{1/3}(1 - \frac{2}{3}y - \frac{5}{9}y^2 + \mathcal{O}(y^3)), \quad (4.15)$$
$$= T_0(1 - \frac{2}{3}y - \frac{5}{9}y^2 + \mathcal{O}(y^3)),$$

where $T_0$ is the Hawking temperature at $\Omega = 0$. The dimensionless ratio $\Omega/T_H$ can be expanded to the second order in $y$ as well

$$(\frac{\Omega}{T_H})^2 = -\frac{2^{4/3}\pi}{3}(\mu g)^{1/3}\sinh\beta(1 + y + \frac{14}{9}y^2 + \mathcal{O}(y^3)), \quad (4.16)$$
$$= \frac{4\pi^2}{9}y(1 + y + \frac{14}{9}y^2 + \mathcal{O}(y^3)).$$

Using the above expression and definitions (4.8), one can write $y$ in terms of the ratio $\Omega/T_H$ up to forth order as follows

$$y = \frac{9}{4\pi^2}(\frac{\Omega}{T_H})^2 - 2\frac{81}{16\pi^4}(\frac{\Omega}{T_H})^4 + \dots \quad (4.17)$$

# 5. Perturbing the R-Charged AdS$_4$ Black Holes

## 5.1 Review of the Reissner-Nordström Black Hole Perturbation Theory

What follows is a lightning review of the perturbation theory of the 4-dimensional Einstein-Maxwell system with a cosmological constant. We closely follow [13]. Coordinates are labeled as $(t, \phi, r, \theta) = (0, 1, 2, 3)$. We follow the mostly minus signature convention for the spacetime metric. Perturbations are assumed to be generically non-stationary but axially symmetric. The most general non-stationary, axially symmetric perturbation of an arbitrary 4-dimensional spacetime can be parameterized as follows

$$ds_4^2 = e^{2\nu}dt^2 - e^{2\psi}(d\phi - \omega dt - q_r dr - q_\theta d\theta)^2 - e^{2\mu_r}dr^2 - e^{2\mu_\theta}d\theta^2. \tag{5.1}$$

It can be shown that the linearized perturbations fall into two distinct decoupled classes. One set, called "polar perturbations", consists of $\delta F_{02}, F_{03}, F_{23}, \delta\nu, \delta\mu_r, \delta\mu_\theta$ while the other set called "axial perturbations", includes $F_{01}, F_{12}, F_{13}, \omega, q_r, q_\theta$, where $F_{ab}$ denotes the Maxwell field strength. We use $\delta$ in front of a fluctuation, whenever the corresponding fluctuation has a non zero background. We will not be considering polar perturbations here since the relevant perturbations to the viscosity computations fall into the *axial perturbations* class. The equations of motion governing perturbations are most easily written in the tetrad basis. The explicit form of the tetrad we use here is given by

$$\begin{aligned}
e_\mu^{\widehat{0}} &= (e^\nu, 0, 0, 0), \\
e_\mu^{\widehat{1}} &= (-\omega e^\psi, e^\psi, -q_r e^\psi, -q_\theta e^\psi), \\
e_\mu^{\widehat{2}} &= (0, 0, e^{\mu_r}, 0), \\
e_\mu^{\widehat{3}} &= (0, 0, 0, e^{\mu_\theta}).
\end{aligned} \tag{5.2}$$

The hatted indices are flat. Note that all the indices refer to the tetrad basis (5.2) unless otherwise mentioned.

## 5.2 The Axial Perturbation Equations of Motion

The axial class of perturbation equations come from the following components of the Einstein and Maxwell's equations

- (ab)=(12) and (13) components of the Einstein equations,

- $\nu = \phi$ component of the Maxwell equations,

- Bianchi identities written for $(\phi, t, r)$ and $(\phi, t, \theta)$ permutations.

The total number of equations sums up to five. There are two more equations which are redundant. In what follows, comma denotes ordinary derivative with respect to the corresponding coordinate. The explicit form of the axial equations of motion for the background (4.4) are written as follows

$$(rf^{1/2}F_{01})_{,r} + (H^2 rf^{-1/2}F_{12})_{,0} = 0, \tag{5.3}$$

$$(rf^{1/2}F_{01})_{,\theta} + (H^2 r^2 F_{13})_{,0} = 0, \tag{5.4}$$

$$(H^2 rf^{-1/2}F_{01})_{,0} + (rf^{1/2}F_{12})_{,r} + (F_{13})_{,\theta} = H^2 r^2 F_{02}Q_{02}. \tag{5.5}$$

where $Q_{0\mathcal{A}} = \omega_{,\mathcal{A}} - q_{\mathcal{A},0}$ and $Q_{\mathcal{AB}} = q_{\mathcal{A},\mathcal{B}} - q_{\mathcal{B},\mathcal{A}}$ and $\mathcal{A}, \mathcal{B} = 1, 2, 3$. Taking derivatives with respect to $r, \theta$ and $t$ in equations (5.3), (5.4) and (5.5) yields

$$(rf^{1/2}F_{01})_{,r\,,r} + (H^2 r f^{-1/2} F_{12})_{,0\,,r} = 0, \tag{5.6}$$

$$(rf^{1/2}F_{01})_{,\theta\,,\theta} + (H^2 r^2 F_{13})_{,0\,,\theta} = 0, \tag{5.7}$$

$$(H^2 r f^{-1/2} F_{01})_{,0\,,0} + (rf^{1/2}F_{12})_{,r\,,0} + (F_{13})_{,\theta\,,0} = H^2 r^2 F_{02} Q_{02,0}. \tag{5.8}$$

From (5.3) we have

$$\left[ f H^{-2}(rf^{1/2}F_{01})_{,r} \right]_{,r} + (rf^{1/2}F_{12,0})_{,r} = 0. \tag{5.9}$$

Utilizing (5.6), (5.7), (5.8) and (5.9) we obtain

$$\left[ H^{-2}f(rf^{1/2}F_{01})_{,r} \right]_{,r} + \frac{f^{1/2}}{H^2 r}(F_{01})_{\theta,\theta} - rH^2 f^{-1/2}(F_{01})_{,0\,,0} = -H^2 r^2 F_{02} Q_{02,2}. \tag{5.10}$$

Now, let us turn to the Einstein equations. From the (12) and (13) components of the Einstein equations, we obtain

$$R_{12} = -\frac{1}{2} e^{-2\psi - \nu - \mu_\theta} \left[ (e^{3\psi + \nu - \mu_r - \mu_\theta} Q_{32})_{,\theta} - (e^{3\psi - \nu - \mu_r + \mu_\theta} Q_{02})_{,0} \right] \tag{5.11}$$
$$= -2F_{01}F_{20}.$$

Using eq(4.4) and eq(5.11) we get

$$\frac{f^{-1/2}}{H^2 r^3} \left[ r^2 f Q_{32,\theta} - H^4 r^4 Q_{02,0} \right] = -4F_{10}F_{02}. \tag{5.12}$$

The component $R_{13}$ yields

$$(r^2 f Q_{23}), r = (H^4 r^2 f^{-1}) Q_{03,0}. \tag{5.13}$$

Equations (5.12) and (5.13) simplify to

$$\frac{1}{H^4 r^4} Q_{,\theta} = -(\omega_{,r} - q_{r,0})_{,0} + \frac{4}{H^2 r} f^{1/2} F_{10} F_{02}, \tag{5.14}$$
$$\frac{f}{H^4 r^2} Q_{,r} = (\omega_{,\theta} - q_{\theta,0})_{,0},$$

where $Q = r^2 f(q_{r,\theta} - q_{\theta,r})$. Since $\partial_t$ and $\partial_\theta$ are Killing directions of the unperturbed background, a typical fluctuation will have the following form

$$\xi(t, r, \theta) = \xi(r) e^{i\sigma t + i q \theta}, \tag{5.15}$$

where $\xi$ denotes a typical fluctuation. [4]

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[4] "$q$" appearing in (5.15) is dimensionless. Therefore to compare our results with [4], one needs to replace the dimensionless $q$ by $q/2g$.

Eliminating $\omega$ from (5.14) leads to

$$\partial_r \left[ \frac{f}{H^4 r^2} Q_{,r} \right] + \frac{1}{H^4 r^4} \partial_\theta^2 Q - \frac{1}{r^2 f} \partial_t^2 Q = \frac{4}{H^2 r} f^{1/2} F_{02} F_{10,\theta} . \tag{5.16}$$

Now let us return to equation(5.10). Using (5.14), one has

$$\left[ H^{-2} f (r f^{1/2} F_{01})_{,r} \right]_{,r} + \frac{f^{1/2}}{H^2 r} (F_{01})_{\theta,\theta} + (\sigma^2 r H^2 f^{-1/2} - 4 r F_{02}^2 f^{1/2}) F_{01} = \frac{F_{02}}{H^2 r^2} Q_{,\theta} . \tag{5.17}$$

Substituting the assumed form for the fluctuations (5.15), one obtains the following pair of equations

$$\left[ H^{-2} f (r f^{1/2} F_{01})_{,r} \right]_{,r} - \frac{q^2 f^{1/2}}{H^2 r} (F_{01}) + (\sigma^2 r H^2 f^{-1/2} - 4 r F_{02}^2 f^{1/2}) F_{01} = i \frac{q F_{02}}{H^2 r^2} Q,$$

$$\partial_r \left[ \frac{f}{H^4 r^2} Q_{,r} \right] - \frac{q^2}{H^4 r^4} Q + \frac{\sigma^2}{r^2 f} Q = -i \frac{4q}{H^2 r} f^{1/2} F_{02} F_{01}. \tag{5.18}$$

It is also useful to work out the second order differential equation satisfied by the fluctuation $\omega$. In order to do so, the Einstein equation corresponding to $R_{01}$ needs to be written down

$$R_{01} = -\frac{1}{2} e^{-2\psi - \mu_r - \mu_\theta} \left[ (e^{3\psi - \nu - \mu_r + \mu_\theta} Q_{20})_{,r} + (e^{3\psi - \nu + \mu_r - \mu_\theta} Q_{30})_{,\theta} \right] \tag{5.19}$$

$$= 2 F_{02} F_{12}.$$

Using the AdS black hole background fields in (4.4) and simplifying the resulting expressions, one is led to

$$\frac{f^{1/2}}{H^4 r^3} \left( [H^4 r^4 (q_{r,0} - \omega_{,r})]_{,r} + [H^4 r^2 f^{-1} (q_{\theta,0} - \omega_{,\theta})]_{,\theta} \right) = -4 F_{02} F_{12}. \tag{5.20}$$

Notice that utilizing the tetrad basis definitions in (5.2), the spacetime $F_{01}$ and $F_{02}$ are given by

$$\mathcal{F}_{01} = r f^{1/2} F_{01}, \tag{5.21}$$

$$\mathcal{F}_{02} = F_{02},$$

where $\mathcal{F}$ denotes the curved spacetime $F$. Using the convenient gauge where $q_r = 0$, (5.18) leads to

$$\frac{d}{dr} \left[ \frac{f}{H^2} \frac{d}{dr} \mathcal{F}_{01} \right] - \frac{q^2}{H^2 r^2} \mathcal{F}_{01} + \frac{\sigma^2 H^2}{f} \mathcal{F}_{01} = i \frac{\sigma H^2 r^2}{u^2} F_{02} \omega_{,r} . \tag{5.22}$$

Rewriting equation (5.20) in the gauge $q_r = 0$ gives

$$-(H^4 r^4 \omega_{,r})_{,r} + H^4 r^2 f^{-1} (q_{\theta,0,\theta} - \omega_{,\theta,\theta}) = -\frac{4 F_{02} F_{12}}{f^{1/2}} H^4 r^3. \tag{5.23}$$

Simplifying the above equation utilizing definitions given in (4.8), we will obtain

$$(H^4 \omega')' - \frac{2}{u} H^4 \omega' - \frac{H^4}{A(H^4 - u^3)} (\sigma q q_\theta + q^2 \omega) = \frac{4 H^4 R_0^2}{u^3 f^{1/2}} F_{02} F_{12}, \tag{5.24}$$

where "prime" refers to $d/du$ and $A = 4g^2 R_0^4$. Using (5.14) and (5.3) combined with the gauge condition and (4.8), one ends up with

$$\frac{A}{H^4 R_0^4}(H^4 - u^3)iqq_{\theta,r} = -i\sigma\omega_{,r} + \frac{4}{H^2 r}f^{1/2}F_{01}F_{02}, \qquad (5.25)$$

$$(rf^{1/2}F_{01})_{,r} = i\sigma H^2 r f^{-1/2}F_{12}.$$

Changing the coordinate system to "$u$", equation (5.22) is written as follows

$$\frac{d}{du}\left[\frac{H^4 - u^3}{H^2}\frac{d}{du}\mathcal{F}_{01}\right] + \frac{1}{A}\Big(\frac{\sigma^2 R_0^4 H^2}{A(H^4 - u^3)} - \frac{q^2}{H^2}\Big)\mathcal{F}_{01} = -i\frac{\sigma\mu\sinh\beta R_0^2}{A}\omega_{,u}. \quad (5.26)$$

Denoting $h = H^4\omega'$ and using (5.24) and (5.25) one ends up with a second order ODE for $h$ which has the following form in "$u$" coordinates

$$h'' - \left(\frac{(u^3 H^{-4})'}{1 - u^3 H^{-4}} + \frac{2}{u}\right)h' + \left(\frac{2}{u^2} + \frac{2}{u}\frac{(u^3 H^{-4})}{1 - u^3 H^{-4}}\right)h = \qquad (5.27)$$

$$\left[\frac{4\sigma^2 H^6 R_0^2}{A^2(H^4 - u^3)^2}F_{02} + \frac{4H^4 R_0^2}{Au^2(H^4 - u^3)}\Big(\frac{q^2 u^2}{H^2 R_0^4} - \frac{\sigma^2 H^2}{f}\Big)F_{02}\right]A_1 + \frac{4u^2 H^2 f}{A(H^4 - u^3)R_0^2}F_{02}'A_{1,u},$$

where, $\mathcal{F}_{01} = -i\sigma A_1 = -i\sigma A_{\phi}$ using the fact that $\partial_t$ is a killing direction and remembering that we are considering axially symmetric perturbations.

## 6. Solving The Coupled System of ODEs

### 6.1 Singularity Structure, Boundary Conditions

The system of coupled differential equations (5.26) and (5.27) forms the fundamental set of equations to be solved. As is clear, these ODEs are singular at $u = u_0$ where $u_0$ is the horizon location. In order to isolate the singularity at $u = u_0$, we substitute the following ansatz into the above ODEs

$$A_1 = (u_0 - u)^\gamma P(u), \qquad (6.1)$$

$$h = (u_0 - u)^\nu F(u).$$

The regularity condition, in addition to the incoming boundary condition for the fluctuations $F(u)$ and $P(u)$ at $u = u_0$, will fix the values of $\gamma$ and $\nu$ (as will be computed later) where $u_0$ is the horizon radius. Substituting the ansatz (6.1)into (5.27) and (5.26) gives

$$F(u)'' + \mathcal{P}(u)F(u)' + \mathcal{Q}(u)F(u) = \mathcal{R}(u)P(u) + \mathcal{S}(u)P(u)',$$

$$P(u)'' + \mathcal{U}(u)P(u)' + \mathcal{V}(u)P(u) = \mathcal{W}(u)F(u),$$

where

$$\mathcal{P}(u) = -\left(\frac{2\nu}{u_0 - u} + \frac{(u^3 H^{-4})'}{1 - u^3 H^{-4}} + \frac{2}{u}\right), \qquad (6.2)$$

$$\mathcal{Q}(u) = \left[ \frac{\nu(\nu-1)}{(u_0-u)^2} + \frac{\nu}{u_0-u} \left( \frac{(u^3 H^{-4})'}{1-u^3 H^{-4}} + \frac{2}{u} \right) + \frac{2}{u^2} + \frac{2}{u} \frac{(u^3 H^{-4})'}{1-u^3 H^{-4}}, \right.$$
$$\left. -\frac{q^2}{A(H^4-u^3)} + \frac{\sigma^2 H^4 R_0^4}{A^2(H^4-u^3)^2} - \frac{4H^2 R_0^4}{Au^2(H^4-u^3)} F_{02}^2 \right],$$

$$\mathcal{R}(u) = \left[ \frac{4\sigma^2 H^6 R_0^2}{A^2(H^4-u^3)^2} F_{02} \right.$$
$$\left. + \frac{4H^4 R_0^2}{Au^2(H^4-u^3)} \left( \frac{q^2 u^2}{H^2 R_0^4} - \frac{\sigma^2 H^2}{f} \right) F_{02} - \frac{4\gamma u^2 H^2 f}{A(H^4-u^3)(u_0-u)R_0^2} F_{02}' \right],$$

$$\mathcal{S}(u) = \frac{4u^2 H^2 f}{A(H^4-u^3)R_0^2} F_{02}',$$

$$\mathcal{U}(u) = -\left( \frac{2\gamma}{u_0-u} + \frac{3u^2-4H^3 H'}{H^4-u^3} + 2\frac{H'}{H} \right),$$

$$\mathcal{V}(u) = \left[ \frac{\gamma(\gamma-1)}{(u_0-u)^2} + \frac{\gamma}{u_0-u} \left( \frac{3u^2-4H^3 H'}{H^4-u^3} + 2\frac{H'}{H} \right) + \frac{\sigma^2 H^4 R_0^4}{A^2(H^4-u^3)^2} - \frac{q^2}{A(H^4-u^3)} \right],$$

$$\mathcal{W}(u) = \frac{\mu \sinh\beta R_0^2}{AH^2(H^4-u^3)}.$$

As mentioned earlier, $\nu$ and $\gamma$ can be computed by demanding regularity for functions $F(u)$ and $P(u)$ at $u = u_0$. $\nu$ and $\gamma$ are thus given by the following expressions

$$\gamma = \nu = \pm i \frac{\sigma R_0^2}{3A} \left( 1 + \frac{2}{3} y + y^2 + \mathcal{O}(y^3) \right). \tag{6.3}$$

Note that the above expression can only be trusted to second order in $\sinh\beta$. The minus sign corresponds to the incoming boundary condition at the horizon and, according to the prescription for calculating Minkowskian retarded Green's functions, is the right boundary condition.

## 6.2 Solving the System in Power Series, Domain of Convrgence

In this subsection, we find the solution to the system (6.2) in a series expansion around $u = u_0$. The aim will be to see if the radius of convergence of the series is large enough to include the point $u = 0$, where one is actually interested in calculating the pole structure of the Minkowskian Green's functions. As is obvious from the ODEs (6.2), there exist 4 singular points: $u = 0$, $u = u_0$, $u = \infty$, and $u = -R_0^2/(\mu \sinh^2\beta)$. Normally the radius of convergence of a series solution as viewed on the complex plane of $u$, extends all the way to the next neighboring singularity. For small values of $\beta$, which is what we are considering here, the point $u = -R_0^2/(\mu \sinh^2\beta)$ will be well outside the convergence circle centered around $u = u_0$ and encompassing $u = 0$. So there appears to be no obstruction to continue the expansion to $u = 0$. Before presenting the solution, let us repackage our coefficients

$$\mathcal{Q}(u) = \left[ \frac{\gamma(\gamma-1)}{(u_0-u)^2} + \frac{\gamma}{u_0-u} \left( \frac{(u^3 H^{-4})'}{1-u^3 H^{-4}} + \frac{2}{u} \right) + \frac{2}{u^2} + \frac{2}{u} \frac{(u^3 H^{-4})'}{1-u^3 H^{-4}}, \right. \tag{6.4}$$
$$\left. -\frac{Q^2}{(H^4-u^3)} + \frac{S^2 H^4}{4(H^4-u^3)^2} - \frac{4xu^2 \sinh^2\beta}{H^2(H^4-u^3)} \right],$$

$$\mathcal{R}(u) = \frac{1}{R_0^4} \left[ 4Q^2 x \frac{u^2 \sinh\beta}{H^4 - u^3} - 8\gamma x \frac{u \sinh\beta}{(u_0 - u)H} \right],$$

$$\mathcal{S}(u) = 8x \frac{u \sinh\beta}{H R_0^4},$$

$$\mathcal{V}(u) = \left[ \frac{\gamma(\gamma-1)}{(u_0 - u)^2} + \frac{\gamma}{u_0 - u} \left( \frac{3u^2 - 4H^3 H'}{H^4 - u^3} + 2\frac{H'}{H} \right) + \frac{S^2 H^4}{4(H^4 - u^3)^2} - \frac{Q^2}{(H^4 - u^3)} \right],$$

$$\mathcal{W}(u) = \frac{R_0^4 \sinh\beta}{H^2(H^4 - u^3)},$$

where $x = \mu/R_0^2, S = \sigma/(g\sqrt{x}), Q = q/\sqrt{x}$. In the hydrodynamic limit, we will be interested only in expansions of the functions $F$ and $P$ at most to 3rd order in $S$ and $Q$. We need to keep terms proportional to $S$ and $Q^2$ and nothing else. This is because of the fact that diffusion phenomena always involve two derivatives with respect to the spatial dimensions, while there is only one derivative with respect to time.

### 6.3 Numeric-Symbolic Solution

In this section, we present our series solution to the coupled system of ODE's for the gravitational and gauge fluctuations.

Let us digress for a moment and focus on how many integration constants one should expect in the solution. We have two second order ODE's, which means there are four integration constants. Two of the four are fixed by requiring the regularity condition for $F(u)$ and $P(u)$ at $u = u_0$. The remaining two integration constants get fixed by imposing boundary condition at the boundary of AdS, i.e., at $u = 0$. Starting from the series solution ansatz

$$F(u) = \sum_{i=0}^{\infty} f_i (u - u_0)^i, \tag{6.5}$$

$$P(u) = \sum_{i=0}^{\infty} p_i (u - u_0)^i,$$

our plan will be to solve for $f_i = f_i(S, Q, y)$ and $p_i = p_i(S, Q, y)$ up to a desired order "N", as a function of the two remaining integration constants (which will turn out to be $f_0$ and $p_0$). Here $y = (9/4\pi^2)(\Omega/T_H)^2 - 2(81/16\pi^4)(\Omega/T_H)^4 + \mathcal{O}((\Omega/T_H)^6)$ is the combination introduced in subsection 4.2. These series coefficients will be further expanded to the first order in S and second order in Q which are the only relevant terms in the hydrodynamic limit

$$f_i(S, Q, y) = \Phi_{i0}(y) + \Phi_{i1}(y)S + \Phi_{i2}(y)Q^2, \tag{6.6}$$

$$p_i(S, Q, y) = \Pi_{i0}(y) + \Pi_{i1}(y)S + \Pi_{i2}(y)Q^2.$$

We further expand $\Phi_{ki} = \Phi_{ki}(y)$ and $\Pi_{ki} = \Pi_{ki}(y)$ in powers of $y$

$$\Phi_{ki}(y) = \phi_{ki0} + \phi_{ki1}y + \phi_{ki2}y^2, \tag{6.7}$$

$$\Pi_{ki}(y) = \pi_{ki0} + \pi_{ki1}y + \pi_{ki2}y^2.$$

The interpretation of the indices is clear. In order to keep our notation simple, we have dropped the explicit dependence of $f_i$ and $p_i$ (and consequently all other expansion coefficients) on $f_0$ and $p_0$. Now all we are required to do will be to compute the coefficients $\phi_{kil}$ and $\pi_{kil}$.

At this stage, we need to impose boundary conditions. To do this, we will have to use the perturbation equation (5.24). Remembering that

$$h = H^4\omega' = (u_0 - u)^\gamma F(u), \tag{6.8}$$

and taking the $u \to 0$ limit of (5.24) give rise to

$$(H^4\omega')'|_{u\to 0} - \frac{2}{u}H^4\omega'|_{u\to 0} - \frac{1}{A}(\sigma q q_\theta^0 + q^2\omega^0) = 0, \tag{6.9}$$

where superscript "0" refers to the boundary values of the fluctuations $q_\theta$ and $\omega$ at the boundary of the spacetime. Notice that the full solution for $F(u)$ must go to zero at $u = 0$ in order for $F(u)$ to be a regular solution of (5.24) [5] . Thus (6.9) can be rewritten as

$$-(H^4\omega')'|_{u\to 0} = F_{10}(S, Q, y)f_0 + F_{11}(S, Q, y)p_0 = \frac{1}{A}(\sigma q q_\theta^0 + q^2\omega^0), \tag{6.10}$$

where, we have substituted the series solution for $F$. We have also taken into account and explicitly indicated the fact that, all the series coefficients are expressed as a function of $S, Q, y$ as well as $f_0$ and $p_0$. Similarly, taking the $u \to 0$ limit of (6.1) gives

$$P(u)|_{u\to 0} = P_{10}(S, Q, y)f_0 + P_{11}(S, Q, y)p_0 = A_1^0, \tag{6.11}$$

where $A_1^0$ refers to the boundary value of $A_1$.

Equations (6.10) and (6.11) provide us with a system of two linear equations for two unknown $f_0$ and $p_0$ which fixes the integration constants $f_0$ and $p_0$ in terms of the boundary values $q_\theta^0$ and $\omega^0$.

The diffusion constant denoted by $\mathcal{D}$ is the location of the $G_{tx,tx}(q, \sigma, y)$ pole which is ultimately related to the shear viscosity through the relation

$$\mathcal{D} = \frac{\eta}{\epsilon + p}. \tag{6.12}$$

The pole is given by the dispersion relation

$$\sigma = -i\mathcal{D}q^2. \tag{6.13}$$

One can easily convince oneself that the location of the desired pole is the zero of the determinant of a 2 by 2 matrix $\Gamma$ made out of $P_{10}$, $P_{11}$, $F_{10}$ and $F_{11}$

$$\Gamma = \begin{pmatrix} P_{10} & P_{11} \\ F_{10} & F_{11} \end{pmatrix}.$$

---

[5]In our series solution $F(u{=}0) = \mathcal{O}(y^3)$, which is zero since we have only kept up to two powers of $y$ at every step of our computations.

The condition $det(\Gamma) = 0$ gives rise to the following pole structure

$$S = -i\lambda(y)Q^2 + \mathcal{O}(y^3). \tag{6.14}$$

In order to compute $\mathcal{D}$, we need to switch to dimensionful quantities. Remembering that $S = \sigma/(g\sqrt{x})$, $Q = q/(\sqrt{x})$, one can write (6.14) as

$$\frac{\sigma}{(2\pi T_0)/3} = -i\lambda(y)(\frac{q}{(2\pi T_0)/3})^2, \tag{6.15}$$

where $T_0$ denotes the Hawking temperature at zero chemical potential. [6] Comparing to the dispersion relation (6.13), one deduces

$$\mathcal{D} = \frac{3\lambda(y)}{2\pi T_0}. \tag{6.16}$$

In fact, $\lambda = \lambda(y)$ is the quantity we calculate numerically as an expansion in $y$.

## 7. Calculating $\eta/s$

We proceed to calculate the shear viscosity using the relation (6.12). Conformal symmetry implies $\epsilon = 2p$, leading to

$$\eta = 3p\mathcal{D} = \frac{9p\lambda(y)}{2\pi T_0}, \tag{7.1}$$

where we have used $\mathcal{D} = 3\lambda(y)/(2\pi T_0)$ from the previous chapter. Note that in the grand canonical ensemble one has

$$p = -\frac{\partial \Xi_{M2}}{\partial V}, \tag{7.2}$$

where $\Xi_{M2} = E - TS - J\Omega$ is the Gibbs free energy for the spinning membrane. The Gibbs free energy for M2 brane with four angular momentum turned on, can be easily calculated [14]. Expanding to fourth order in powers of $\Omega/T_H$, one gets

$$p = -\frac{\Xi_{M2}}{V} = \frac{2^{7/2}\pi^2}{3^4}N^{3/2}T_H^3\left[1 + \frac{9 \times 4}{8\pi^2}(\frac{\Omega}{T_H})^2 + \frac{27}{16\pi^4}(\frac{\Omega}{T_H})^4 + \ldots\right]. \tag{7.3}$$

Similarly, the entropy density is given by

$$s = -\frac{1}{V}\frac{\partial \Xi_{M2}}{\partial T} = \frac{2^{7/2}\pi^2}{3^3}N^{3/2}T_H^2\left[1 + \frac{9 \times 4}{24\pi^2}(\frac{\Omega}{T_H})^2 - \frac{27}{48\pi^4}(\frac{\Omega}{T_H})^4 + \ldots\right]. \tag{7.4}$$

Assuming

$$\lambda(y) = \lambda_0 + \lambda_1 y + \lambda_2 y^2 + \mathcal{O}(y^3), \tag{7.5}$$

---

[6]Note that in order to compare our results at leading order with [4], we need to send $q \rightarrow q/(2g)$.

the ratio $\eta/s$ becomes

$$
\begin{aligned}
\frac{\eta}{s} &= \frac{3\lambda_0}{2\pi}\left(1 + \frac{9}{4\pi^2}(\frac{2}{3} + \frac{\lambda_1}{\lambda_0})(\frac{\Omega}{T_H})^2 + \right. \\
&\quad \left. + \frac{81}{16\pi^4}(-\frac{5}{9} - \frac{4}{3}\frac{\lambda_1}{\lambda_0} + \frac{\lambda_2}{\lambda_0})(\frac{\Omega}{T_H})^4 + \ldots\right) \\
&= \frac{3\lambda_0}{2\pi}(1 + \zeta_2(\frac{\Omega}{T_H})^2 + \zeta_4(\frac{\Omega}{T_H})^4 + \ldots).
\end{aligned}
\tag{7.6}
$$

As we mentioned in the previous chapter, $\lambda$ is the quantity we compute numerically. Thus, calculating $\lambda$ will provide us with the corrections to the ratio at zero chemical potential i.e., $1/(4\pi)$.

Surprisingly, the following numerical analysis presented here illustrates that the second and fourth order corrections to the $\eta/s$ ratio asymptotes to zero as we keep more and more terms in the Taylor expansion for F and $A_\phi$. This clearly signals a saturation of the bound even in the presence of a non zero chemical potential.

The ratio at zero chemical potential has been calculated before to be $1/(4\pi)$ [4], therefore we expect to get $\lambda(y = 0) = 1/6$ at the leading order. Of course, this only serves as a consistency check to assure us that the numerics have been done carefully. Below, we have plotted $\lambda_0 = \lambda(y = 0)$ and $\zeta_2$ versus $N$, the number of terms in the Taylor expansions for F and $A_\phi$ i.e., (6.5). The coefficient $\zeta_4$ also runs to zero rather quickly with increasing N as seen from the figure 3. Figure 4 is the plot of shear viscosity versus $y$ for N= 30.
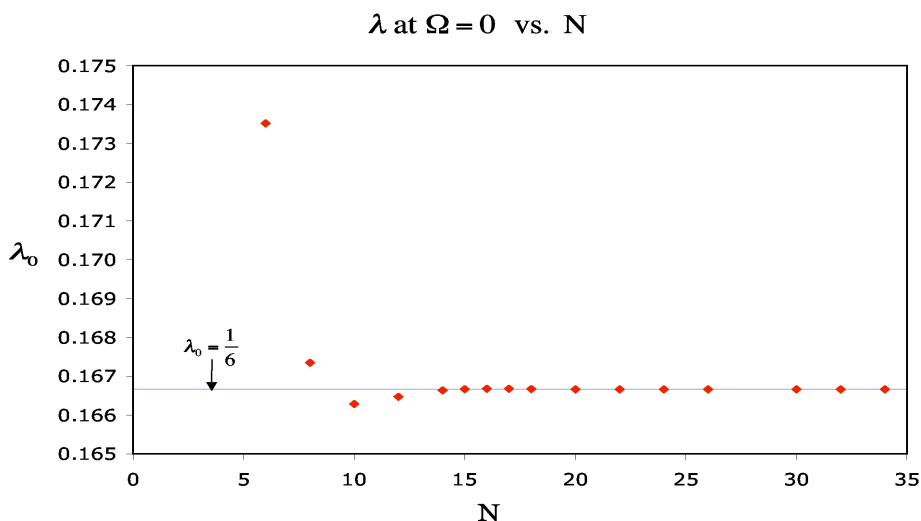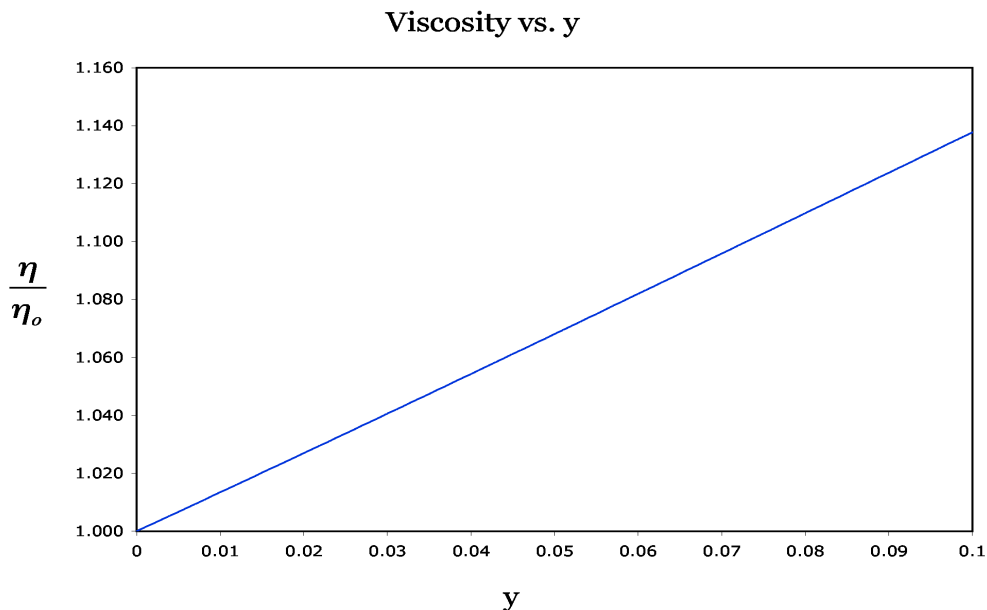


**Figure 1:** $\lambda$ plotted Vs. N for $\Omega = 0$.
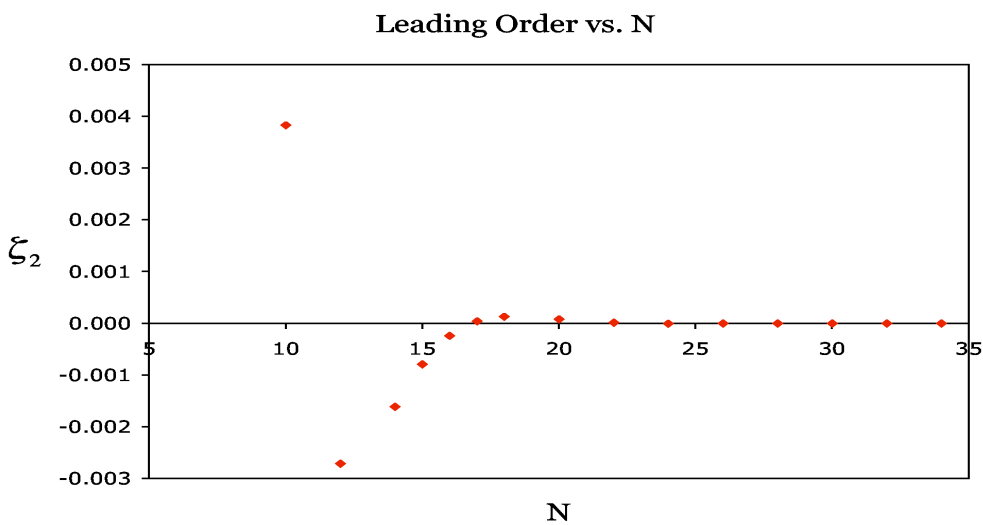
**Figure 2:** $\zeta_2$ plotted Vs. N.
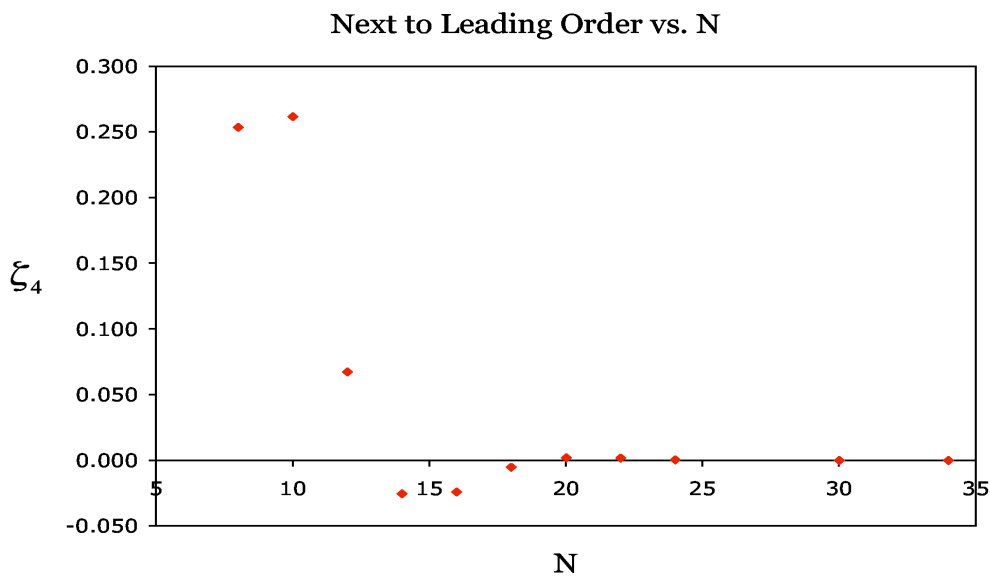


**Figure 3:** $\zeta_4$ plotted Vs. N.

**Figure 4:** Viscosity Vs. $y$, plotted only for small values of $y$. $\eta_0$ is the shear viscosity at $y = 0$.

## 8. Conclusion and outlook

The result presented in this paper satisfies the viscosity bound conjecture proposed by Policastro, Son and Starinets [5]. The leading finite chemical potential corrections to the viscosity itself turn out to be positive. The ratio $\eta/s$ remained unchanged up to fourth order in $\Omega/T_H$ signalling the saturation of the bound. An interesting question is why shear viscosity increases even though the system is literally at "infinite coupling". Another way of putting this is to say, what could make an "infinitely coupled" system , "less infinitely coupled" (in order for the viscosity to increase)! One could speculate that the reason behind this enhancement in shear viscosity might lie in some screening effect for the color interaction mediator at finite chemical potential. At non zero chemical potential i.e., when the number of various particle species carrying different R-charge is imbalanced, the "gluons" mass receives correction from the chemical potential which could result in screening . This screening of the color charge weakens the effective interactions in the plasma which ultimately leads to a bigger mean free path. When this paper was being written, I became aware of two other works [20], [21] in preparation on AdS$_5$ with similar results. While for the membranes (which were studied in this paper), no gauge theory description exists, for the AdS$_5$ system there is a gauge theory living on the world-volume of D3-branes. Using the conformal invariance of N$=$ 4 SYM theory, only based on dimensional grounds, one could argue that the gluons mass receives corrections which are proportional

to $\mu$, where $\mu$ is the chemical potential [7]. A similar situation occurs in perturbative QCD. In this case, one could speculate with more confidence that the screening effect may be the true reason for an increase in the shear viscosity of a hot gauge theory plasma at finite chemical potential. Of course, the theory under consideration here is strongly coupled so perturbative notions may no longer be applicable.

It was argued and quantified by Karch [15] that the conjectured viscosity bound is connected to Bousso's Generalized Covariant Entropy Bound (GCEB). Given such an interesting interrelation, one could reinterpret the viscosity bound as a *non-gravitational* and *empirical* window to the realm of *quantum gravity*. It turns out that the viscosity bound is exactly what matter is required to obey in order for gravity to modify the light-sheets (motion of the viscous fluid results in a stress $T_{ij}$-generated curvature) to prevent the GCEB from a catastrophic violation. The current formulation of the GCEB suffers from a number of problems including "the species problem". The species problem is the simple statement that the entropy of a system of field(s) confined in a region of space can grow simply by increasing the number of particle species while keeping the total energy fixed. This would lead to a violation of the GCEB. An exciting question [16] would be to ask whether violating the GCEB through the species problem, for instance, would lead to a violation of the viscosity bound. To address this question one has to perform similar calculations as outlined in the present paper for supergravity duals to the gauge theories with large $N_f$ [17] (see also [19]) at "finite temperature". A zero temperature realization was considered in [19], where the field theory corresponding to the localized D2-D6 intersection is an $\mathcal{N} = 4$, $d = 3$ super Yang Mills gauge theory coupled to $N_6$ hypermultiplets in the fundamental of the gauge group, where $N_6$ is the number of D6-branes. $N_6/N_2$ is kept fixed while $N_2, N_6 \gg 1$. The full supergravity background (i.e., D6-flavor branes including back-reaction) has been worked out in [18], where the fact that uplifted D6-branes to M-theory has a Taub-NUT space component proves to be helpful. The non-extremal version of [18] is not yet known and it seems like a daunting task to carry out. It would be interesting to find the background at least in the form of an expansion series in the vicinity of the horizon. Extensions of similar sets of computations would teach us a lot about whether there is a violation of the viscosity bound at large $N_f$.

Needless to say, finding analytic solutions to the coupled differential equations here would be invaluable as it could reveal analytic structure of the viscosity as a function of $\Omega/T_H$.

There is no $G_N$ in the statement of the bound. This simply indicates that the conjecture is a statement about quantum mechanical matter without any reference to gravity. So it is natural to expect the existence of a proof or counterexample for the bound which only involves weakly gravitating quantum physics. The relevance of gravity seems to be solely a consequence of the fact that in order to get down to the saturation limit of the bound, one is required to go to exceedingly high values of the coupling which, in the light of AdS/CFT, is mapped to the physics of highly gravitating objects, i.e., black holes in Anti de Sitter space.

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[7]$\mu$ has dimension mass. Note that, this effect is on top of the usual finite temperature corrections.

## Acknowledgments

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